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Numerical Evaluation of Airy Functions with Complex Arguments

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We present two methods for the evaluation of Airy functions of complex argument. The first method is accurate to any desired precision but is slow and unsuitable for fixed-precision languages. The second method is accurate to double precision (12 digits) and is suitable for programming in a fixed-precision language such as FORTRAN. The first method uses the symbolic manipulation language Maple to evaluate either the Taylor series expansion or an asymptotic expansion of each function. The second method extends an idea of J. C. P. Miller to the complex plane. It uses the first method to obtain a grid of points in the complex plane where the functions are known to high precision and then uses Taylor series from these base points. The resulting algorithm is accurate and efficient. © 1992 Academic Press, Inc.

1. INTRODUCTION

In this paper we present two methods for the numerical evaluation of the Airy functions with complex arguments; these are the solutions of Airy's equation

$$y'' - zy = 0. \tag{1.1}$$

The calculation of the Airy functions with real arguments has been carried out by many authors; a good description of the early literature can be found in Miller [1]. Recently a differential equation method has been applied to the numerical evaluation of the Airy, Pearcy, and swallowtail canonical integrals by Connor *et al.* [2], again with real arguments. Schulten *et al.* [3] considered the complex Airy functions. They based their algorithm mainly on numerical evaluation of Stieltjes-type integral representations of the functions using generalized Gaussian quadrature. Near the origin of the complex plane they used a power series expansion.

The present paper differs from the paper by Schulten *et al.* [3] in that both methods presented here are essentially simpler. We demonstrate in this paper that the use of modern computer algebra languages allows us to overcome the limitations that caused these simpler methods to be passed over previously in favour of more complicated techniques.

The Airy functions appear in the solution of several problems in fluid mechanics, geophysics, and atomic physics. We now briefly discuss some of these. More details can be found in the cited references.

For plane Couette flow it can be shown (Drazin and Reid [4]) that the Orr–Sommerfeld equation for linear stability has the form

$$(\varepsilon^3(D^2-\alpha^2)-\eta)(D^2-\alpha^2)\phi=0 \tag{1.2}$$

with boundary conditions

$$\phi = D\phi = 0 \qquad \text{at} \quad \eta=-1-c, \eta=1-c.$$

Here D stands for $d/d\eta$ and $\varepsilon=(\imath\alpha R)^{-1/3}$. This can be interpreted as an eigenvalue problem for the growth factor c for given values of the Reynolds number R and the wavelength α.

If we let

$$y=(D^2-\alpha^2)\phi$$

and

$$z=\varepsilon^{2/3}(\alpha^2+\varepsilon^{-2}\eta)$$

we see that Eq. (1.2) reduces to the Airy equation (1.1). Thus it is clear that the four linearly independent solutions for ϕ can be expressed in terms of integrals containing the two Airy functions. Recently a similar approach has been used by Hooper and Boyd [5, 6] who developed similar equations in their studies of shear flow instability.

In magnetotelluric data inversion one tries to deduce the conductivity structure of the earth from the measurements of the impedance function. For one-dimensional problems it can be shown that under reasonable assumptions the impedance $c(k, z)$ is given by

$$(D^2-k^2\sigma(z))\, c(k,z)=0, \tag{1.3}$$

where $D=d/dz$ and k is the wavenumber of the incoming

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signal and $\sigma(z)$ is the unknown conductivity profile. The boundary conditions on $c(k, z)$ are

$$Dc = -1 \qquad \text{at} \quad z = 0, c \to 0 \text{ as } z \to \infty.$$

The unknown conductivity profile $\sigma(z)$ is to be determined when the data $c(k, 0)$ is measured for different values of k. For more details, see, e.g., Coen *et al.* [7].

Usually as part of the inversion procedure the direct problem with $\sigma(z)$ known must be solved. If a layered model of the earth is used, $\sigma(z)$ is usually either constant or proportional to z. In the latter case Eq. (1.3) becomes the Airy equation. This approach has been used, for example, by Kao and Rankin [8] in a study of a three-layer earth.

The numerical evaluation of the Airy functions is also required in semi-classical descriptions of atom–atom collision. Bieniek [9] discusses uniform JWKB amplitudes and phases for turning-point problems in this field.

The outline of the paper is as follows. In the next section we summarize briefly some of the well-known properties of the Airy functions that we require in our analysis. In Section 3 we present our first method. It consists of using the symbolic manipulation language Maple [10] to evaluate the Taylor series representations of the Airy functions for complex z to the required accuracy. For large $|z|$ we use asymptotic formulae. While these algorithms are computationally expensive the programs are fairly simple. In Section 4 we develop a fixed-precision algorithm in FORTRAN, based on the methods of Miller [1].

2. PROPERTIES OF AIRY FUNCTIONS

We here summarize the properties of the Airy functions which we use below. We use the notation of Olver [11]. Airy's differential equation is

$$y'' - zy = 0. \tag{2.1}$$

Any solution $y(z)$ of (2.1) is entire and has the symmetry that $y(\omega z)$ and $y(\bar{\omega} z)$ are also solutions, where $\omega = \exp(2\pi\imath/3)$. Further,

$$y(z) + \omega y(\omega z) + \bar{\omega} y(\bar{\omega} z) = 0. \tag{2.2}$$

Two linearly independent solutions which are real when z is real are $Ai(z)$ and $Bi(z)$, which satisfy the initial conditions (Olver [11])

$$Ai(0) = \frac{3^{-2/3}}{\Gamma(2/3)}, \qquad Ai'(0) = \frac{-3^{-1/3}}{\Gamma(1/3)}$$

and

$$Bi(0) = \frac{3^{-1/6}}{\Gamma(2/3)}, \qquad Bi'(0) = \frac{3^{1/6}}{\Gamma(1/3)}.$$

We note at this point that, since $Ai(\omega z)$ and $Ai(\bar{\omega} z)$ are solutions of (2.1), they are necessarily linearly dependent on $Ai(z)$ and $Bi(z)$. Explicitly (Abramowitz and Stegun [12], we have

$$Ai(\omega z) = \tfrac{1}{2} e^{\imath\pi/3} \{Ai(z) - \imath Bi(z)\} \tag{2.3}$$

and

$$Ai(\bar{\omega} z) = \tfrac{1}{2} e^{-\imath\pi/3} \{Ai(z) + \imath Bi(z)\}. \tag{2.4}$$

The connection formulae (2.2)–(2.4) are used extensively in the sections to follow. We will also require the following two asymptotic series for $Ai(z)$ (Olver [11]):

$$Ai(z) \sim \frac{1}{2} \pi^{-1/2} z^{-1/4} e^{-\xi} \sum_{s=0}^{\infty} (-1)^s \frac{u_s}{\xi^s} \qquad \text{valid when} \quad |\text{Arg } z| < \pi, \tag{2.5}$$

where $\xi = \frac{2}{3} z^{3/2}$ in principal value, and

$$Ai(-z) \sim \pi^{-1/2} z^{-1/4} \left(\cos\left(\xi - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \times \frac{u_{2s}}{\xi^{2s}} + \sin\left(\xi - \frac{\pi}{4}\right) \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{\xi^{2s+1}} \right) \tag{2.6}$$

valid when $|\text{Arg } z| \leqslant 2\pi/3$, where $u_s = [(2s+1)(2s+3) \cdots (6s-1)]/(216)^s s!$, which together suffice to cover the whole plane. Asymptotic formulae for Bi are also available but are not used in this paper.

The Taylor series for $Ai(z)$ and $Bi(z)$ about zero may be written as (Abramowitz and Stegun [12])

$$Ai(z) = Ai(0) f(z) + Ai'(0) g(z) \tag{2.7}$$

and

$$Bi(z) = Bi(0) f(z) + Bi'(0) g(z), \tag{2.8}$$

where

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{3^n \Gamma(n+1/3)}{\Gamma(1/3)(3n)!} z^{3n}$$

and

$$g(z) = z + \sum_{n=1}^{\infty} \frac{3^n \Gamma(n+2/3)}{\Gamma(2/3)(3n+1)!} z^{3n+1}.$$

For large positive z, $f(z)$ and $g(z)$ both grow exponentially, but $Ai(z)$ decays exponentially. We shall thus have a

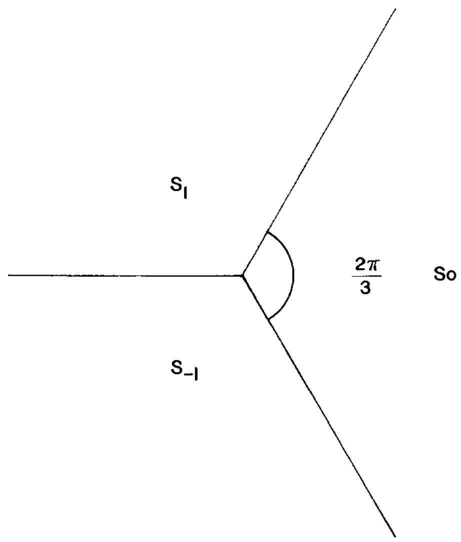

FIG. 1. Regions in which $Ai(\omega z)$, $Ai(z)$, or $Ai(\bar{\omega} z)$ are "recessive" or exponentially subdominant.

cancellation problem to deal with in the calculation of $Ai(z)$ from $f(z)$ and $g(z)$.

We note at this point that the solutions $Ai(z)$, $Ai(\bar{\omega} z)$, and $Ai(\omega z)$ can be recessive, where a recessive solution is one that decays exponentially, as $|z| \to \infty$, along a ray from the origin relative to the other linearly independent solutions, i.e.,

$Ai(z)$ is recessive in the region
$$S_0 = \{z \mid |\operatorname{Arg} z| < \pi/3\},$$
$Ai(\bar{\omega} z)$ is recessive in the region
$$S_1 = \{z \mid \pi/3 < \operatorname{Arg} z < \pi\},$$
$Ai(\omega z)$ is recessive in the region
$$S_{-1} = \{z \mid -\pi < \operatorname{Arg} z < -\pi/3\}.$$

These regions are illustrated in Fig. 1.

$Bi(z)$ is not recessive in any S_i, while Ai is recessive only in S_0. (Olver [11]). Thus, in view of the linear dependencies (2.3) and (2.4), the mathematically linearly independent $Ai(z)$ and $Bi(z)$ are *numerically* linearly *dependent* for (even moderately) large values of $|z|$ in the regions S_1 and S_{-1}. Therefore linearly independent solutions that are numerically satisfactory are in S_0 the pair $\{Ai(z), Bi(z)\}$, in S_1 the pair $\{Ai(z), Ai(\bar{\omega} z)\}$, and in S_{-1} the pair $\{Ai(z), Ai(\omega z)\}$ (Olver [11]).

3. FIRST METHOD OF EVALUATION

Modern symbolic manipulation languages (e.g., Maple [10]) have the ability to calculate with exact rationals of very large size, or to calculate with floating-point numbers of very high precision, limited only by the memory of the machine used. Either of these two facilities enables us to use a very simple brute-force approach to the evaluation of the Airy functions. This approach, while expensive in terms of computer time, is very simple to implement and enables us to get accurate benchmarks from which we may derive a more efficient fixed-precision subroutine. Indeed, for some purposes, the symbolic manipulation implementation is sufficient in itself.

3.1. *Summation of Taylor Series for Small and Moderate* $|z|$

Any solution $y(z)$ of (2.1) can be written $y(z) = \sum_{k=0}^{\infty} Y_k$, where the Y_k are determined by the recurrence relation

$$Y_k = \frac{Y_{k-3} z^3}{k(k-1)}, \qquad k \geqslant 3, \tag{3.1}$$

with appropriate starting values.

In particular the solutions $y = f(z)$ and $y = g(z)$ have initial conditions

$$F_0 = 1, \qquad F_1 = 0, \qquad F_2 = 0 \tag{3.2}$$

and

$$G_0 = 0, \qquad G_1 = z, \qquad G_2 = 0. \tag{3.3}$$

So $f(z) = \sum_{k=0}^{\infty} F_k$ and $g(z) = \sum_{k=0}^{\infty} G_k$, where F_k and G_k also obey (3.1). Note that $|G_k| \leqslant |z|\, |F_k|$ for all k. We use Maple to evaluate these series.

Note that only every third term in each series is nonzero: F_{3k} in the series for f, and G_{3k+1} in the series for g. Thus

$$F_{3k} = \frac{3^k \Gamma(k+1/3)\, z^{3k}}{\Gamma(1/3)(3k)!}$$

and

$$G_{3k+1} = \frac{3^k \Gamma(k+2/3)\, z^{3k+1}}{\Gamma(2/3)(3k+1)!}.$$

If we give z to our Maple program as an exact rational number, possibly also with some algebraic or transcendental constant (e.g., $\sqrt{2}$, π, $\Gamma(1/3)$), we get the exact sum of the truncated power series, which we may then convert to decimal form (at which point the cancellation problem manifests itself, so we must ensure that enough decimals are taken here to obviate the problem). If we use the very high precision capability from the start (say with 50 or 60 digits) the program is somewhat faster than with exact rational arithmetic and just as accurate. We need only take enough terms in the Taylor series to ensure that the truncation error is small enough to give a good relative error (or absolute error if we are near a zero). To facilitate the choice of the number of terms, an analysis of the Taylor series was done to find a bound on the truncation error.

To bound the truncation error we start with the recurrence relation (3.1) and the definition of the Taylor series sum for either $S=f(x)$ or $S=g(z)$:

$$S=\sum_{k=0}^{\infty} Y_k=\sum_{k=0}^{n} Y_k+\sum_{k=n+1}^{\infty} Y_k. \tag{3.4}$$

If we take n so $Y_n \neq 0$, then by the construction of the series for f or g, $Y_{n+1}=Y_{n+2}=0$. Therefore,

$$\begin{aligned} S&=\left(\sum_{k=0}^{n} Y_k\right)+Y_{n+3}+Y_{n+6}+Y_{n+9}+\cdots \\ &=\left(\sum_{k=0}^{n} Y_k\right)+Y_{n+3}\left(1+\frac{z^3}{(n+6)(n+5)}\right. \\ &\quad\left.+\frac{z^6}{(n+9)(n+8)(n+6)(n+5)}+\cdots\right). \end{aligned}$$

Note that for $k>6$, we have $(n+k)(n+k-1)\geqslant(n+5)^2$, which can be used in the equation above to show that the truncation error $T=Y_{n+3}+Y_{n+6}+Y_{n+9}+\cdots$ satisfies

$$\begin{aligned} |T| &\leqslant |Y_{n+3}| \sum_{k=0}^{\infty}\left|\frac{z^{3k}}{(n+5)^{2k}}\right| \\ &=|Y_{n+3}| \sum_{k=0}^{\infty} \hat{r}_{n+1}^{k} \end{aligned}$$

which converges if $\hat{r}_{n+1}=|z^3|/(n+5)^2<1$.
Then $|T|\leqslant|Y_{n+3}|/(1-\hat{r}_{n+1})$ because the series giving the upper bound is geometric.

We then find that

$$\begin{aligned} |\Delta Ai(z)|&=|Ai(z)-\hat{A}i(z)| \\ &=\left|Ai(z)-\left(Ai(0)\left(\sum_{k=0}^{n} F_k\right)+Ai'(0)\left(\sum_{k=0}^{n} G_k\right)\right)\right| \\ &\leqslant (Ai(0)-|z|\;Ai'(0))\frac{|F_{n+1}|}{1-r_{n+1}} \qquad \text{if} \quad r_{n+1}<1, \end{aligned}$$

where we change notation slightly so that

$$r_{n+1}=|z|^3/(3n+5)^2.$$

This gives us a relative error bound, if further $|\hat{A}i(z)|>|\Delta Ai(z)|$ (and hence $|Ai(z)|>0$), of

$$\frac{|\Delta Ai(z)|}{|Ai(z)|}\leqslant\frac{(Ai(0)-|z|\;Ai'(0))|F_{n+1}|}{\left(\begin{array}{l}(1-r_{n+1})\;|\hat{A}i| \\ \quad -(Ai(0)-|z|\;Ai'(0))\;|F_{n+1}|\end{array}\right)} \tag{3.5}$$

and everything on the right-hand side is easily computable.

An exactly similar analysis gives

$$\frac{|\Delta Bi(z)|}{|Bi(z)|}\leqslant\frac{(Bi(0)+|z|\;Bi'(0))\;|F_{n+1}|}{\left(\begin{array}{l}(1-r_{n+1})\;|\hat{B}i(z)| \\ \quad -(Bi(0)+|z|\;Bi'(0))\;|F_{n+1}|\end{array}\right)}. \tag{3.6}$$

A Maple routine has been written to calculate $Ai(z)$, $Ai'(z)$, $Bi(z)$, and $Bi'(z)$ by Taylor series and to return error estimates based on the above formulae. Note that the idea of numerical linear dependence plays a smaller role in the very high precision context: one need only take enough figures and the mathematical linear independence is evident.

3.2. *Summation of Asymptotic Series for Large* $|z|$

It is clear that the cost of computing $Ai(z)$ to a given precision increases with increasing $|z|$. At some point we are better off using the asymptotic formulae (2.5) and (2.6) together with the connection formulae (2.3) or (2.4) to calculate the desired quantities. It remains to choose a value of $|z|$ to switch from the Taylor series to the asymptotic series. Call this value R.

The choice of the change-over radius R depends on the precision desired. Olver [11] gives sharp truncation error bounds for the asymptotic formulae for (2.5) and (2.6) (presented in the next section) and using these we find that for a relative error of less than 10^{-16} we cannot use the asymptotic formulae if $|z|<10$. For other precisions Table I exhibits our results.

Thus, e.g., for double precision we would use the Taylor series with (at most) 52 nonzero terms inside $|z|\leqslant 10$, while in $|z|>10$ twenty-three terms in the asymptotic series (either (2.5) or (2.6) depending on Arg z) gives at least the same accuracy. For much larger $|z|$, fewer asymptotic terms are required; for $|z|$ much smaller than 10, fewer Taylor terms are required to reach this precision. For $|z|<10$, the asymptotic series cannot achieve the desired accuracy.

To obtain truncation error bounds for the asymptotic series we use the asymptotic formula (2.5) in the range $|\text{Arg}\, z|\leqslant 2\pi/3$, but not in $|\text{Arg}\, z|<\pi$ since the accuracy degrades near the Stokes line on the negative real axis. Olver [11] gives the error bound for

$$Ai(z)=\pi^{-1/2}z^{-1/4}e^{-\xi}\left(\sum_{s=0}^{n-1}(-1)^s\frac{u_s}{\xi^s}+\eta\right)$$

TABLE I

Precision	R	Nonzero Taylor terms in $\lvert z\rvert \leqslant R$	Asymptotic terms $\lvert z\rvert \geqslant R$
10^{-6}	5	19	12
10^{-16}	10	52	23
10^{-23}	12	58	50
10^{-33}	15	78	65

as

$$|\eta| \leqslant 2 \cdot \chi(n) \cdot \frac{u_n}{|\xi|^n} \cdot \exp\left\{\frac{5}{36} \cdot \chi(1) \cdot |\xi|^{-1}\right\}, \tag{3.7}$$

where

$$\chi(n) = \pi^{-1/2} \frac{\Gamma(n/2+1)}{\Gamma((n+1)/2)} = \left(\frac{n}{2}\right)^{1/2} + O(n^{-1/2})$$

grows only slowly with n.

If $n = 25$ and $|z| > 10$, we have

$$|\eta| \leqslant 12.75 \cdot \frac{u_n}{|\xi|^n} \leqslant 15 \cdot \frac{u_n}{|\xi|^n}$$

or $\leqslant 15$ times the modulus of the first neglected term. We use the asymptotic formula (2.6) in the remaining ranges $2\pi/3 \leqslant \operatorname{Arg} z \leqslant \pi$ and $-\pi \leqslant \operatorname{Arg} z \leqslant -2\pi/3$, and a similar result obtains

$$Ai(-z) = \pi^{-1/2} z^{-1/4} \left\{\cos\left(\xi - \frac{\pi}{4}\right) \times \left[\sum_{s=0}^{n} (-1)^s \frac{u_{2s}}{\xi^{2s}} + \eta_1\right] + \sin\left(\xi - \frac{\pi}{4}\right)\left[\sum_{s=0}^{n} (-1)^s \frac{u_{2s+1}}{\xi^{2s+1}} + \eta_2\right]\right\}$$

and both η_1 and η_2 satisfy

$$|\eta_i| \leqslant 2\chi(2n+2) \frac{u_{2n+2}}{|\xi|^{2n+2}} \cdot \exp\left\{\frac{5}{36} \chi(1) \, |\xi|^{-1}\right\}. \tag{3.8}$$

This gives us practical truncation error bounds for the computation of the asymptotic series for $Ai(z)$. Note that $Bi(z)$ may be calculated in $|z| > R$ by using one of (2.3) or (2.4) and calculating $Ai(z)$ and either $Ai(\omega z)$ or $Ai(\bar{\omega} z)$ by the asymptotic formulae given. Similar formulae are used for the evaluation of $Ai'(z)$.

Both the Taylor-series method and the asymptotic-series method were programmed in Maple. For a given value of z, the program selects the appropriate strategy and then returns Ai and Bi and bounds on the error based on one of the formulae derived above.

4. EVALUATION IN FIXED-PRECISION WITH FORTRAN

Once we have a reliable method (however expensive in computing time) for the computation of $Ai(z)$, $Ai'(z)$, $Bi(z)$, and $Bi'(z)$, we may investigate more efficient, fixed-precision algorithms. Here we develop a method, also based on Taylor series, for efficient evaluation of the Airy functions. We note that the main numerical difficulty in calculating Ai is the catastrophic cancellation along the real axis and, to a lesser extent, along the rays $\operatorname{Arg}(z) = \pm\pi/3$, which also gives difficulty in the computation of Bi.

We first briefly show the nature of the problem. Consider the Taylor series for $y(z)$ based at $z = c$. Then

$$y(z) = \sum_{k=0}^{n} Y_k + \sum_{k=n+1}^{\infty} Y_k,$$

where the second sum may be made as (mathematically) small as we please by taking n large enough. However, in fixed precision we do not actually calculate $y_n = \sum_0^n Y_k$, but rather

$$\hat{y}_n = \sum_{k=0}^{n} Y_k(1 + \rho_k \mathbf{u}),$$

where $\mathbf{u}$ is the machine epsilon, defined as the smallest machine representable number such that, when stored, $1 + \mathbf{u}$ is different from 1, and where the ρ_k are growth factors due to propagation of roundoff errors in the recurrence relation (Gautschi [14]). We have

$$\begin{aligned} |y - \hat{y}_n| &= |y - y_n + y_n - \hat{y}_n| \\ &\leqslant |y - y_n| + |y_n - \hat{y}_n| \\ &\leqslant |y - y_n| + \left|\sum_0^n Y_k \rho_k \mathbf{u}\right| \\ &\leqslant |y - y_n| + \mathbf{u}\left(\sum_0^n |Y_k|\right) \max_{0 \leqslant k \leqslant n} |\rho_k|, \end{aligned}$$

where the second term on the right is a bound for the roundoff error. Therefore, if $\rho_n = \max_{0 \leqslant k \leqslant n} |\rho_k|$, the relative roundoff error

$$C = \left|\frac{y_n - \hat{y}_n}{y}\right|$$

satisfies

$$C \leqslant \frac{\rho_n \mathbf{u} \sum_{k=0}^{n} |Y_k|}{|y|} \quad \text{if} \quad y \neq 0 \tag{4.1}$$

and we see that if $|y|$ is small and $\sum |Y_k|$ is large, C could be large. This occurs on the real axis for $y = Ai$, when $\sum |Y_k|$ is like Bi and grows exponentially and occurs for both Ai and Bi along $|\operatorname{Arg} z| = \pi/3$. When $y(z)$ is near a zero, then we also have difficulty unless $\sum |Y_k|$ is small, which is usually impossible. The problem is unavoidable in this case and an absolute tolerance must be used here instead.

A similar analysis for the asymptotic series gives the following bound for the cancellation error:

$$C \leqslant \left(\sum_{s=0}^{n} \frac{u_s}{|\xi|^s} \right) \Bigg/ \left| \sum_{s=0}^{n} \frac{(-1)^s u_s}{\xi^s} \right| \mathbf{u} \quad \text{for} \quad |\operatorname{Arg} z| \leqslant \frac{2\pi}{3}, \tag{4.2}$$

and

$$C \leqslant \frac{\left(\begin{array}{l} 3 \cdot (|\cos(\xi - (\pi/4))| \sum_{s=0}^{n} (u_{2s}/|\xi|^{2s}) \\ \quad + |\sin(\xi - (\pi/4))| \sum_{s=0}^{n} (u_{2s+1}/|\xi|^{2s+1})) \end{array} \right)}{\left(\begin{array}{l} |\cos(\xi - (\pi/4)) \sum_{s=0}^{n} (u_{2s}/\xi^{2s}) \\ \quad + \sin(\xi - (\pi/4)) \sum_{s=0}^{n} (u_{2s+1}/\xi^{2s+1})| \end{array} \right)} \mathbf{u}$$

for

$$|\operatorname{Arg} z \leqslant 2\pi/3|. \tag{4.3}$$

These problems are usually cured by taking n small, so that $\sum |Y_k|$ is not too large. This "low-order" approach means that smaller mesh sizes are necessary in order to keep the truncation error small.

However, "high-order" is desirable for efficiency, and it is still possible if we choose our expansion point to make $|y|$ *relatively large*, not small. This means that we must expand about the *right* end point on a real interval for a calculation of Ai; calculations based on the left end point are stable for the calculation of Bi. The basic idea, due to Miller [1] and used by him in the computation of $Ai(x)$ for real x, is to pick a base point that allows the desired component of the general solution to the ODE to grow exponentially.

4.1. *Evaluation Using a Triangular Grid*

Thus, since $Ai(z)$ decays as we move towards $+\infty$ in the sector $|\operatorname{Arg} z| < \pi/3$, we must expand about a base point which is closer to $+\infty$ than z. Then $Ai(z)$ will be exponentially large at the desired point, compared with the value of $Ai(z)$ at the base point. This makes the factor $|y|$ in the error bound for C large, and thus the cancellation error is small.

This proves successful on the real axis for Ai, though not completely so because the signs in the Taylor series still alternate like $++-++-\cdots$ and the cancellation problem still exists, albeit in a much reduced form. The authors are not aware of any other work in which this idea is applied in the complex plane, but the idea also works in any region where Ai and Bi both exhibit exponential behaviour. Luckily, in the sector $|\operatorname{Arg} z| < \pi/3$ they both do so, and from this sector we can stably evaluate them anywhere in the complex plane by using the connection formulae. However, it is not so clear just where to put the base points in the complex plane. We have implemented the scheme below.

Note that along $\operatorname{Arg} z = \pi/3$, we do not have exponential growth in either Ai or Bi, but rather oscillation. Thus we *must* use the "low-order" idea along this ray. However, because $|y|$ decays only algebraically as $|z| \to \infty$ along this ray, the cancellation problem is not as severe to begin with.

To account properly for the numerical difficulties caused by exponential decay, we first decompose the wedge $\{z \mid \operatorname{Arg} z < \pi, \ |z| < R\}$ into triangles and near-triangles. Figure 2a shows the decomposition used for the double precision calculations. When evaluating $Ai(z)$ or $Bi(z)$, we first determine which triangle contains z. Within a triangle we follow the strategy illustrated in Fig. 2b. To calculate $Ai(z)$, we use the Taylor expansion about the lower right corner, so that we are calculating $Ai(z)$ in a direction

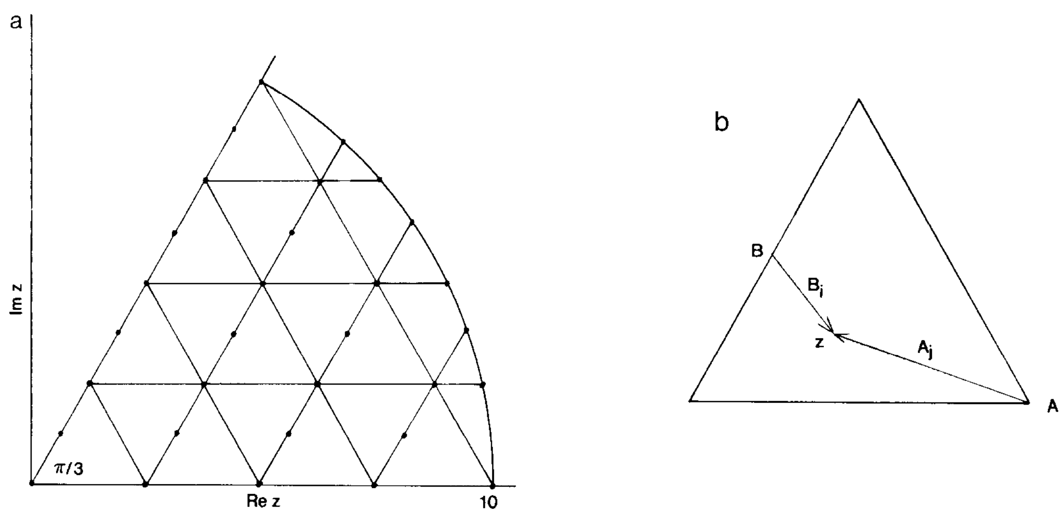

FIG. 2. a. Mesh decomposition of the region $|z| \leqslant 10$, $0 \leqslant \operatorname{Arg} z \leqslant \pi/3$ for the evaluation of the Airy functions in complex double precision. b. Base points used for the calculation of Airy functions at z by Taylor series. The point A is used for the Taylor series for Ai, and the point B for Bi.

leading to exponential *growth* of the *desired* component. Likewise, to calculate $Bi(z)$ we use the Taylor expansion about the upper left point, again so that we calculate in a direction giving exponential growth of the desired component.

4.2. *Truncation Error Analysis for the Modified Taylor Series Method*

One advantage of the Taylor series is that we can obtain a useful (for diagnostic purposes) bound on the truncation error. We saw earlier a bound on the cancellation error for the Taylor series. After the truncation error bound for the Taylor series is given, we shall derive similar results for the cancellation error in the asymptotic series.

We compute $y(z) = y(c + z - c) = y(c + h)$ via

$$\begin{aligned} y = y_n + T_n &= \sum_{k=0}^{n} \frac{y^{(k)}(c)}{k!} h^k + \sum_{k=n+1}^{\infty} \frac{y^{(k)}(c)}{k!} h^k \\ &= \sum_{k=0}^{n} A_k + \sum_{k=n+1}^{\infty} A_k, \end{aligned}$$

where $A_k = (y^{(k)}(c)/k!)\, h^k$ can be found using the recurrence relation

$$A_k = \frac{h^2}{k(k-1)} (c \cdot A_{k-2} + A_{k-3} \cdot h), \qquad k \geqslant 3 \tag{4.4}$$

and A_0, A_1 are determined by initial conditions and $A_2 = ch^2 A_0/2$.

Now $T_n = \sum_{n+1}^{\infty} A_k$ satisfies $|T_n| \leqslant \sum_{n+1}^{\infty} |A_k|$ (which also converges), and

$$\begin{aligned} |A_k| &\leqslant \frac{|h|^2}{k(k-1)} (|c|\, |A_{k-2}| + |h|\, |A_{k-3}|) \\ &\leqslant \frac{|h|^2}{(n+1)n} (|c|\, |A_{k-2}| + |h|\, |A_{k-3}|), \qquad k \geqslant n+1 \end{aligned}$$

so if

$$a_k = \frac{|c|\, |h|^2}{n(n+1)} a_{k-2} + \frac{|h|^3}{n(n+1)} a_{k-3}$$

and

$$a_{n-2} = |A_{n-2}|, \qquad a_{n-1} = |A_{n-1}|, \qquad a_n = |A_n|,$$

then $|A_k| \leqslant a_k$ for $k \geqslant n+1$ by induction. Further,

$$a_k = \alpha {\rho_1}^k + \beta {\rho_2}^k + \gamma {\rho_3}^k,$$

where ρ_1, ρ_2, and ρ_3 are the roots of

$$\rho^3 = \frac{|c|\, |h|^2}{n(n+1)} \rho + \frac{|h|^3}{n(n+1)}.$$

Putting $\rho' \cdot |h| \cdot \varepsilon = \rho$, where $\varepsilon^3 = 1/n(n+1)$,

$$(\rho')^3 = 1 + \varepsilon\, |c|\, \rho'$$

and this cubic is obviously numerically stable to solve for small ε. Then

$$a_k \leqslant (|\alpha \rho_1^{n-2}| + |\beta \rho_2^{n-2}| + |\gamma \rho_3^{n-2}|)\, \rho^{k-(n-2)},$$

where $\rho = \max(|\rho_1|, |\rho_2|, |\rho_3|)$, and instead of finding α, β, γ, we solve the more stable linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ \rho_1 & \rho_2 & \rho_3 \\ \rho_2^1 & \rho_2^2 & \rho_3^3 \end{bmatrix} \begin{bmatrix} \alpha \rho_1^{n-2} \\ \beta \rho_2^{n-2} \\ \gamma \rho_3^{n-2} \end{bmatrix} = \begin{bmatrix} |A_{n-2}| \\ |A_{n-1}| \\ |A_n| \end{bmatrix}$$

(which is a system of Vandermonde type). Finally,

$$\begin{aligned} |T_n| \leqslant \sum_{n+1}^{\infty} a_k &\leqslant (|\alpha \rho_1^{n-2}| + |\beta \rho_2^{n-2}| \\ &\quad + |\gamma \rho_3^{n-2}|) \cdot \frac{\rho^3}{1-\rho} \qquad \text{if} \quad \rho < 1 \end{aligned}$$

is our computable bound for the truncation error. This bound is clearly useful only for diagnosis and identification of trouble spots, and for *constructing* the FORTRAN program. For example, we can use this to decide in advance how many terms are necessary in each triangle to achieve the desired accuracy.

4.3. *Stability of the Recurrence Relations*

The error amplification factors ρ_n in Eq. (4.1) come from the propagation of roundoff error through the recurrence relation (4.4). A strict mathematical analysis of this error is beyond the scope of this paper, because (4.4) is a four-term relation. Existing analyses apply only to the three-term case. Olver [13] examines thoroughly the case of the general linear three term recurrence relation. However, to be confident that our results are correct we must know at least if our recurrence relation is stable when evaluated in the forward direction. To this end it is not enough to note that, because of the factorials present in the Taylor series coefficients and the fact that the Taylor series ultimately converges for all h, the recurrence relation is ultimately stable: we need to know also that the relative errors made early in the sequence are not large, because these contribute to the cancellation error (cf. Eq. (4.1)). We note that the ultimate behaviour of the Taylor series coefficients implies that the solutions to the recurrence relation that we are interested in are not "minimal" (Olver [13], Gautschi [14], Wimp [15]) by which is meant that the other linearly independent solutions of the recurrence relation dominate the minimal solution as n goes to infinity, leading to amplification of the roundoff error. If

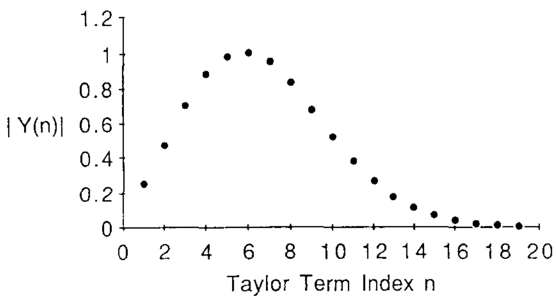

FIG. 3. The magnitude of the complex Taylor series terms for a Taylor stepsize of magnitude 4, which, for emphasis, is larger than any actually used by the program.

our Taylor series coefficients had in fact been minimal, the special technique of backward recursion, also due to Miller [16], would have to be applied. It is of interest to note that the basic idea behind our second algorithm for the evaluation of the Airy functions is precisely the same idea as that used to handle instability of recurrence relations, only we are using it (as indeed Miller [1] did also, in the real case) to evaluate a function on the complex plane instead of the integers.

In the case $c = 0$, the four-term recurrence relation (4.4) can be replaced by a two-term relation, and a stability analysis like that of Gautschi [17] can be carried out explicitly, with the conclusion that the relation is neutrally stable. More simply, we may note that the two-term relation requires only multiplication, and hence is stable. If, however, c is nonzero, as it is for our fixed-precision method, the recurrence relation becomes a four-term one, and, though it is possible in theory to obtain precise information on the three solutions of (4.4), by, for example, the matrix methods mentioned in Olver [13] or Wimp [15], we note that the solution depends nonlinearly on the expansion point c, so we do not expect useful information from such an approach. Instead, numerical experiments have been carried out to investigate the relative error in the Taylor terms $y_n(c)$ that result from perturbations in the initial conditions and in the point of expansion, c. Typical results of these experiments are presented in Fig. 3 and 4. The propagation of errors in the recurrence relation for y_n depends in part on the magnitudes $|y_n|$. If these terms are large, initial errors are amplified by these factors. We see in Fig. 3 the magnitudes of a typical sequence of Taylor series terms, with magnitudes at most $O(1)$, which cause no difficulty. Of course, for large h the height of the "hump" increases, and eventually errors could become serious. For the values of h used by our program (everywhere $|h| \leqslant$ maximum triangle diameter ~ 2.5). This is not a problem. In the graph presented, $|h| = 4$ which is larger than that used in the program, for emphasis. In Fig. 4, we observe the error amplification factors due to a perturbation in c, the base point about which we are Taylor expanding, for the worst case observed. The only values of c that give any trouble lie on or near the ray $\operatorname{Arg}(z) = \pi/3$, where we see that perturbations in c may give relative errors in the series coefficients on the order of 100 times the initial perturbation. This large relative error is due to the fact that the series coefficients themselves oscillate and some terms become small. We note that it is the relative error that is significant, and if the stepsize is large, the cancellation error introduced might also be large. However, the worst such examples observed had amplification factors that were no larger than 100, so we expect to lose no more than two digits of accuracy due to the amplification of error in the recurrence relation.

Thus, practically, the recurrence relation is stable. We note that we have not explored any possible interaction effects between perturbations of c and ongoing roundoff errors, so this investigation cannot be regarded as complete. No further experiments seem justified, however, as no evidence of pathology was seen.

4.4. *Asymptotic Series in Fixed-Precision*

We had no difficulty implementing the asymptotic formulae (2.5) and (2.6), using the symmetry relations (2.3) and (2.4) to cover the entire region $|z| > 10$. Similar accuracy was obtained.

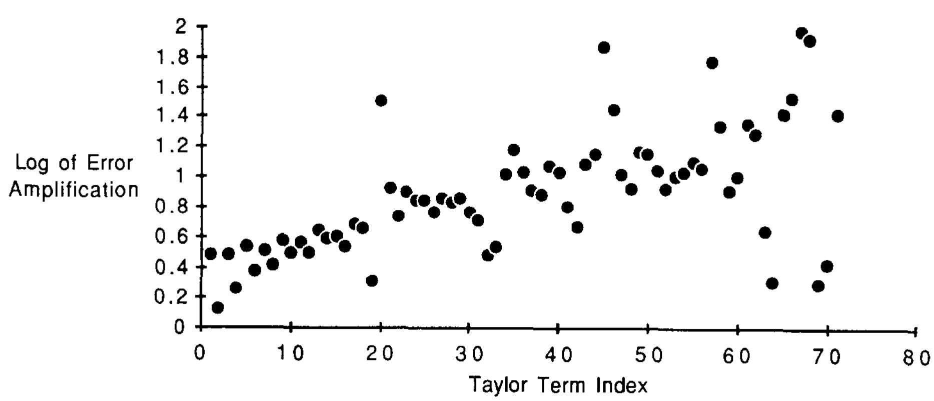

FIG. 4. The relative error amplication $\log_{10}\{10^3\,|(y_n - \tilde{y}_n)/y_n|\}$ caused by a perturbation of size 10^{-3} in the value of "c" (cf. Eq. (4.4)).

TABLE II

Zeros of $Bi(z)$ in Upper Half Plane

Real Part	Imaginary Part
0.977544886731 62E+00	0.21412907060387E+01
0.18967750138953E+01	0.36272917643589E+01
0.26331577393549E+01	0.48554681799798E+01
0.32785312361567E+01	0.59445042811791E+01
0.38658527317333E+01	0.69416922095821E+01
0.44116118748093E+01	0.78718396594866E+01
0.49255293538614E+01	0.87499825412567E+01
0.54139368088077E+01	0.95860969005548E+01
0.58812467539812E+01	0.10387227390304E+02
0.63306885670631E+01	0.11158581226760E+02
0.67647152114495E+01	0.11904144485949E+02
0.71852451085166E+01	0.12627054083962E+02
0.75938143918964E+01	0.13329834712851E+02

5. CONCLUSIONS

The first method, using Maple, is accurate to any desired number of figures. We note that for comparable accuracy the computation time is perhaps two orders of magnitude larger than the computation time for the second method, and with commensurate demands on computer memory as well. Still, for some purposes this is perfectly adequate—for example, in developing and testing other methods.

The second method, based on Miller's idea, is efficient and accurate. We exhibit in Table II the first 12 zeros of $Bi(z)$ in $\pi/3 < \text{Arg}(z) < \pi/2$, found by using Newton's method with the evaluations of $Bi(z)$ and $Bi'(z)$ carried out by the second method.

Since the first method is simple in concept, we are able to give computable error bounds on the computation. The second method allows a conservative error estimate as well.

Future work to find an optimal or near optimal mesh for each of Ai and Bi may be carried out by using the first method to develop an extended-precision implementation of the second algorithm, and then equidistributing the error in the double and single precision versions.

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